CONSTRAINED PARAMETER OPTIMIZATION BY THE METHOD OF EXPLICIT FUNCTIONS: APPLICATION TO THE MESSENGER MISSION

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The solution of a nonlinear system of equations subject to constraints that involve minimizing a scalar performance index is required for many applications, particularly trajectory optimization. Numerical solutions are obtained on a computer by searching the space of independent parameters until the equations of constraint and condition of optimality are satisfied. A constrained parameter search and optimization algorithm based on the method of explicit functions is described. A fundamental equation describing parameter optimization subject to constraints is developed. This equation is used to show the relationship of the method of explicit functions to other optimization methods including Lagrange multipliers and gradient projection. A second-order gradient search algorithm is described. Another algorithm is developed to enable inequality constraints to control the second-order gradient search algorithm. As an example, the first two trajectory correction maneuvers of the MESSENGER mission to Mercury are optimized to minimize propellant consumption.

Introduction

The problem of constrained parameter optimization involves finding the solution of a system of equations that satisfy a number of constraints and minimizes or maximizes some performance criterion which is a scalar measure of the cost of satisfying the constraints. A simple example of a problem of constrained optimization is finding the largest rectangle that will fit inside an ellipse. Except for a few simple examples, problems of constrained optimization are difficult to solve analytically. Numerical solutions may be obtained on a computer with algorithms designed to search the space of independent parameters and find the point within this space that satisfies all the constraints specified and minimizes a scalar performance index that is a given function of the independent parameters.

When properly formulated, an optimization algorithm may be used to solve a wide variety of problems that extend beyond simple parameter optimization. For example, problems of the calculus of variations may be solved by representing a continuously varying control function by a finite set of control parameters and solving for the parameters. The trajectory optimization problem of finding the optimum programmed thrust direction for a low thrust rocket engine may be solved in this fashion.

An optimization algorithm is described[1−4] that solves the problem of constrained optimization by the method of explicit functions. This method was originally devised to minimize propellant expenditure for the Viking mission to Mars. Additional arbitrary constraint functions are adjoined to the given equations of constraint to completely span the space of the independent parameters. The search is performed on these arbitrary parameters to obtain the values of these parameters that minimize the performance criterion. First derivatives of the constraint functions with respect to the independent parameters are used to drive the dependent constraint variables or target variables to satisfy the desired constraints, and second partial derivatives of the minimization criterion with respect to the same independent parameters are used to drive the optimization condition to zero. The search is referred to as a second-order gradient search.

The partial derivatives that are required by the optimization algorithm may be obtained analytically or by finite difference. Analytic partial derivatives are often not pursued because of the difficulty in obtaining the partial derivatives, particularly the second derivatives. A problem with exact second derivative finite difference equations is the large number of function evaluations that are required to compute the derivatives for one iteration. These grow as the square of the number of parameters. Approximate techniques may be used to accelerate the computation of the second derivatives, and a method along the lines suggested by Fletcher-Powell-Davidon is given in References 5 and 6. However, these acceleration techniques generally work well only for the problems they were designed to solve and require modification for specific problems making parameter optimization more of an art than a science.

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Because of non linearity and ill conditioned problems, a second-order gradient search will often diverge. An algorithm is developed to enable inequality constraints to control the search for a solution. Constraining the dependent target variables to an interval permits the optimization algorithm to find a minimum solution within the interval and prevents the search from diverging to a local maximum or inflection point outside the interval. As an example, the first two trajectory correction maneuvers of the MErcury Surface Space ENvironment, GEochemistry, and Ranging (MESSENGER)mission to Mercury are optimized to minimize propellant consumption.

Statement of Problem

A performance index (J) is defined that is a function of N independent variables ($\mathbf{U}$). We also have M equations of constraint ($M < N$) that define the target variables ($\mathbf{Z}_C$), and the equations of constraint are also functions of $\mathbf{U}$. Thus we have

$$J = f(\mathbf{U}) \tag{1}$$

$$\mathbf{Z}_C = g(\mathbf{U}) \tag{2}$$

and

$$\begin{aligned} J &= f(U_1, U_2, U_3, \ldots U_N) \\ Z_{C1} &= g_1(U_1, U_2, U_3, \ldots U_N) \\ Z_{C2} &= g_2(U_1, U_2, U_3, \ldots U_N) \\ &\cdot \quad \cdot \quad \cdot \\ Z_{CM} &= g_M(U_1, U_2, U_3, \ldots U_N) \end{aligned}$$

The problem is to find a $\mathbf{U}^*$ such that

$$\mathbf{Z}_C(\mathbf{U}^*) = \mathbf{C} \tag{3}$$

where $\mathbf{C}$ are constant target parameters and J is a minimum for all $\mathbf{U}$.

Condition for Optimum Solution

A simple method, in principle, for solving the problem of constrained optimization is to solve the equations of constraint (g) for a selected subset of the independent parameters ($\mathbf{U}_C$) and substitute these expressions into the objective function (f), thus reducing the number of unknowns from N to $N - M$; where M is the number of constraint functions. The partial derivative of J with respect to the remaining independent parameters $\mathbf{U}_F$ are obtained and set equal to zero. These equations are solved in conjunction with the equations of constraint. The selection of which independent control parameters to include in $\mathbf{U}_F$ or $\mathbf{U}_C$ is arbitrary. However, the choice may have some effect on performance when a numerical solution is sought.

The method of explicit functions described in this paper carries this concept a step further. In place of the arbitrary selection of control parameters, additional arbitrary constraint functions ($\mathbf{Z}_F$) are defined to bring the total number of $\mathbf{Z}$ parameters to N. The $\mathbf{Z}_F$ functions are not completely arbitrary in that a one to one mapping must exist between $\mathbf{U}$ and $\mathbf{Z}$. At the solution point, any change in $\mathbf{U}$ holding $\mathbf{Z}_C$ constant will increase J. Since a one to one mapping must exist, any unique change in $\mathbf{Z}_F$ holding $\mathbf{Z}_C$ constant will cause a unique change in $\mathbf{U}$ holding $\mathbf{Z}_C$ constant and consequently increase J. Mathematically, setting equal to zero the partial derivative of J with respect to $\mathbf{Z}_F$ holding $\mathbf{Z}_C$ constant is a necessary and sufficient condition for a stationary point, which is a minimum if J is properly defined and $\mathbf{Z}_C$ is properly constrained. As long as there exists a one to one mapping between $\mathbf{U}$ and $\mathbf{Z}$, the same minimum is obtained whether $\mathbf{Z}_F$ or $\mathbf{U}_F$ is selected to minimize J. The performance criterion and augmented equations of constraint are given by

$$\begin{aligned} J &= f(U_1, U_2, U_3, \ldots U_N) \\ Z_{C1} &= g_1(U_1, U_2, U_3, \ldots U_N) \\ Z_{C2} &= g_2(U_1, U_2, U_3, \ldots U_N) \\ &\cdot \quad \cdot \quad \cdot \\ Z_{CM} &= g_M(U_1, U_2, U_3, \ldots U_N) \\ &\cdot \quad \cdot \quad \cdot \\ Z_{FN} &= g_N(U_1, U_2, U_3, \ldots U_N) \end{aligned}$$

and the solution is obtained by solving

$$\mathbf{Z}_C = \mathbf{C} \tag{4}$$

$$\frac{\partial J}{\partial \mathbf{Z}_F} = 0 \tag{5}$$

Observe that the above solution reduces to direct elimination if $\mathbf{Z}_F$ is taken to be identically equal to $\mathbf{U}_F$.

Because of the difficulty in obtaining the inverse functions analytically, direct solution of the above equations is only practical for relatively simple systems of equations. For complex systems, solutions may be obtained by searching using Newton's method. The theory behind techniques currently in use such as Lagrange multipliers and gradient projection follow directly from the method of explicit functions.

The method of explicit functions involves adjoining to the equations of constraint some additional equations that define the parameters $\mathbf{Z}_F$. The $\mathbf{Z}_F$ parameters replace the independent parameters $\mathbf{U}_F$ selected by the method of direct elimination for the purpose of minimizing J. An equation that relates the optimization condition to the independent control parameters, equations of constraint and performance criterion may be obtained by application of the chain rule.

$$\frac{\partial J}{\partial \mathbf{U}} = \frac{\partial J}{\partial \mathbf{Z}} \; \frac{\partial \mathbf{Z}}{\partial \mathbf{U}} \tag{6}$$

The partial derivatives of $\mathbf{Z}$ with respect to the independent parameters $\mathbf{U}$ are contained in a square matrix of dimension N by N. The partial derivatives of J with respect to $\mathbf{U}$ and $\mathbf{Z}$ are row matrices also of dimension N. Partitioning the above matrices separating the $\mathbf{Z}_C$ dependent elements from the $\mathbf{Z}_F$ dependent elements yields

$$\left[\frac{\partial J}{\partial \mathbf{U}}\right] = \left[\begin{array}{cc} \frac{\partial J}{\partial \mathbf{Z}_C} & \frac{\partial J}{\partial \mathbf{Z}_F} \end{array}\right] \left[\begin{array}{c} \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}} \\ \frac{\partial \mathbf{Z}_F}{\partial \mathbf{U}} \end{array}\right] \tag{7}$$

The above partitioned matrices may be factored to further separate those sub matrices dependent on $\mathbf{Z}_C$ from those dependent on $\mathbf{Z}_F$, and after rearranging terms the following equation is obtained:

$$\frac{\partial J}{\partial \mathbf{U}} - \frac{\partial J}{\partial \mathbf{Z}_C}\frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}} = \frac{\partial J}{\partial \mathbf{Z}_F}\frac{\partial \mathbf{Z}_F}{\partial \mathbf{U}} \tag{8}$$

Equation 8 provides a fundamental relationship that may be used to tie together various methods of constrained parameter optimization including the methods of Lagrange multipliers, gradient projection and explicit functions. Comparison of these methods provides insight into which approach may work best depending on the problem.

<u>Method of Lagrange Multipliers</u>

The classic solution of constrained parameter optimization was derived by the eighteenth-century mathematician Joseph Luis Lagrange. This solution is particularly appealing since a choice of independent parameters is not necessary. Referring to Equation 8, at the solution point, the right side is zero because the partial derivative of J with respect to the $\mathbf{Z}_F$ elements must be zero as required by Equation 5:

$$\frac{\partial J}{\partial \mathbf{U}} - \frac{\partial J}{\partial \mathbf{Z}_C} \; \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}} = 0 \tag{9}$$

The terms of Equation 9 may be readily obtained from the equations of constraint and the equation for the performance index with the exception of the partial derivative of J with respect to the $\mathbf{Z}_C$. Lagrange's insight was to make the elements of this term parameters to be solved for in conjunction with the equations of constraint. These parameters are called Lagrange multipliers and are defined by

$$\boldsymbol{\lambda} = -\frac{\partial J}{\partial \mathbf{Z}_C} \tag{10}$$

The sign of the Lagrange multipliers is arbitrary, and it may be conjectured that Lagrange selected the minus sign for convenience. He was certainly aware of Equation 8 but apparently did not consider the right side important since the computer had not been invented in his time. The equations that must be solved to obtain an optimum are thus

$$\mathbf{Z}_C = \mathbf{C} \quad \text{(M equations)} \tag{11}$$

$$\frac{\partial J}{\partial \mathbf{U}} + \boldsymbol{\lambda}\frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}} = 0 \quad \text{(N equations)} \tag{12}$$

The method of Lagrange multipliers requires the solution of M+N equations for N **U** parameters and M Lagrange multipliers. This method is well-suited for obtaining analytic solutions since the equations of constraint need not be solved for the independent **U** parameters as a function of the **Z** parameters. However, the need to solve for the Lagrange multipliers makes numerical solutions[7] more complicated than necessary.

Method of Explicit Functions

The methods of explicit functions and gradient projection use the right side of Equation 8 to obtain a solution and thus avoid the need to solve for Lagrange multipliers. The method of explicit functions requires an equation for the partial derivative of J with respect to $\mathbf{Z}_F$. Application of the chain rule gives

$$\frac{\partial J}{\partial \mathbf{Z}} = \frac{\partial J}{\partial \mathbf{U}} \; \frac{\partial \mathbf{U}}{\partial \mathbf{Z}} \tag{13}$$

The partial derivatives of **U** with respect to the dependent target parameters **Z** are obtained by matrix inversion:

$$\begin{bmatrix} \dfrac{\partial J}{\partial \mathbf{Z}_C} & \dfrac{\partial J}{\partial \mathbf{Z}_F} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial J}{\partial \mathbf{U}} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathbf{Z}_C}{\partial \mathbf{U}} \\ \dfrac{\partial \mathbf{Z}_F}{\partial \mathbf{U}} \end{bmatrix}^{-1} \tag{14}$$

where

$$\frac{\partial \mathbf{U}}{\partial \mathbf{Z}} = \begin{bmatrix} \dfrac{\partial \mathbf{Z}}{\partial \mathbf{U}} \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{\partial \mathbf{Z}_C}{\partial \mathbf{U}} \\ \dfrac{\partial \mathbf{Z}_F}{\partial \mathbf{U}} \end{bmatrix}^{-1}$$

The equations that must be solved to obtain an optimum are the equations of constraint and the equations defined by the last $N - M$ columns of Equation 14:

$$\mathbf{Z}_C = \mathbf{C} \quad \text{(M equations)} \tag{15}$$

$$\frac{\partial J}{\partial \mathbf{Z}_F} = 0 \quad \text{(N} - \text{M equations)} \tag{16}$$

The method of explicit functions requires the solution of N equations for N control parameters **U**. This algorithm is well suited for obtaining numerical solutions on a computer but not for analytic solutions since it involves inversion of a matrix with analytic functions for elements. Observe that the Lagrange multipliers are obtained as a byproduct of Equation 14 (the first M columns).

Method of Gradient Projection

The method of gradient projection[5,8] is a special case of the method of explicit functions. The independent parameters are partitioned into what are referred to as state parameters ($\mathbf{U}_C$) and decision parameters ($\mathbf{U}_F$). The choice between which independent parameters to designate as decision parameters is not unique. The distinction between state and decision parameters is generally only a matter of convenience. However, according to Reference 8, the decision parameters must determine the state parameters through the constraint relations. Expanding Equation 14, separating the $\mathbf{U}_C$-dependent elements from the $\mathbf{U}_F$-dependent elements, yields

$$\begin{bmatrix} \frac{\partial J}{\partial \mathbf{Z}_C} & \frac{\partial J}{\partial \mathbf{Z}_F} \end{bmatrix} = \begin{bmatrix} \frac{\partial J}{\partial \mathbf{U}_C} & \frac{\partial J}{\partial \mathbf{U}_F} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_C} & \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_F} \\ \frac{\partial \mathbf{Z}_F}{\partial \mathbf{U}_C} & \frac{\partial \mathbf{Z}_F}{\partial \mathbf{U}_F} \end{bmatrix}^{-1} \tag{17}$$

The $\mathbf{Z}_F$ constraint relationships have yet to be specified. Depending on the choice of which $\mathbf{U}$ are included in $\mathbf{U}_F$, some reordering of the rows and columns of Equation 14 may be necessary. Also, $\mathbf{U}_F$ will be of dimension $N - M$. Since the selection of the $\mathbf{Z}_F$ equations of constraint is arbitrary, $\mathbf{Z}_F$ may be made identically equal to $\mathbf{U}_F$. Equation 17 then reduces to

$$\begin{bmatrix} \frac{\partial J}{\partial \mathbf{Z}_C} & \frac{\partial J}{\partial \mathbf{Z}_F} \end{bmatrix} = \begin{bmatrix} \frac{\partial J}{\partial \mathbf{U}_C} & \frac{\partial J}{\partial \mathbf{U}_F} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_C} & \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_F} \\ 0 & I \end{bmatrix}^{-1} \tag{18}$$

Performing the indicated matrix inversion yields

$$\begin{bmatrix} \frac{\partial J}{\partial \mathbf{Z}_C} & \frac{\partial J}{\partial \mathbf{Z}_F} \end{bmatrix} = \begin{bmatrix} \frac{\partial J}{\partial \mathbf{U}_C} & \frac{\partial J}{\partial \mathbf{U}_F} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_C}^{-1} & -\frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_C}^{-1} \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_F} \\ 0 & I \end{bmatrix} \tag{19}$$

and

$$\frac{\partial J}{\partial \mathbf{Z}_F} = \frac{\partial J}{\partial \mathbf{U}_F} - \frac{\partial J}{\partial \mathbf{U}_C} \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_C}^{-1} \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_F} = 0 \tag{20}$$

Equation 20 is solved in conjunction with the equation of constraint to obtain an optimum as is done for the method of explicit functions (Equations 15 and 16). Observe that the Lagrange multipliers are obtained as a byproduct from both the method of explicit functions and gradient projection:

$$\boldsymbol{\lambda} = -\frac{\partial J}{\partial \mathbf{Z}_C} = -\frac{\partial J}{\partial \mathbf{U}_C} \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_C}^{-1} \tag{21}$$

Even though the Lagrange multipliers do no enter into the optimal solution, they are useful for determining which bound is appropriate for inequality constraints.

Sample Problem

A sample problem is solved to illustrate the various methods of constrained parameter optimization. Consider an ellipse with semi-major axis a and semi-minor axis b oriented along the Cartesian U_1 and U_2 coordinate axes. The problem is to find the greatest rectangle with sides parallel to the coordinate axes that will fit inside the ellipse. The geometry is illustrated in Figure 1. The equation of constraint describes an ellipse, and the performance criterion is the area of the rectangle. The area in the first quadrant is multiplied by four and assigned a minus sign since we are seeking a maximum:

$$Z_c = \frac{U_1^2}{a^2} + \frac{U_2^2}{b^2} = C = 1 \tag{22}$$

$$J = -4U_1U_2 \tag{23}$$

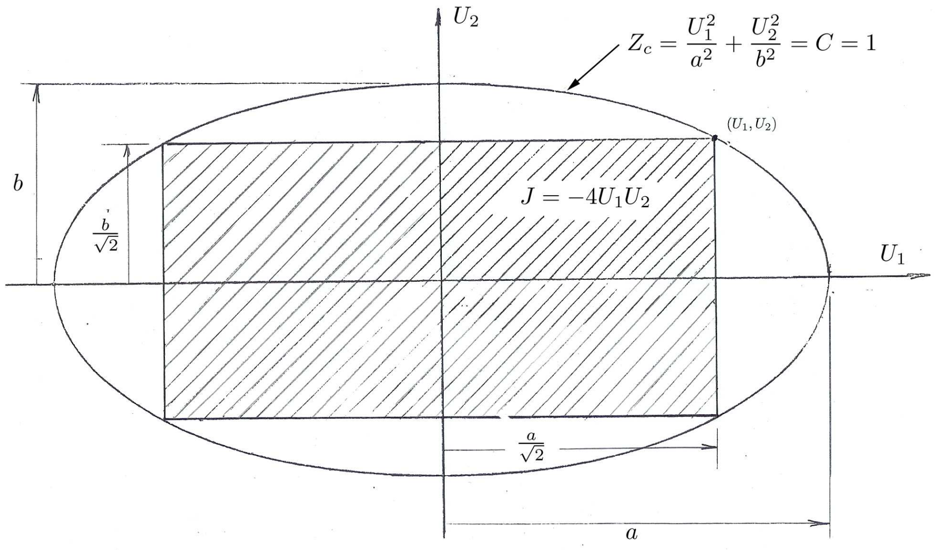

Figure 1 Sample Problem

Solution by Method of Lagrange Multipliers

The method of Lagrange multipliers requires a solution of Equation 12 in conjunction with the equation of constraint (Equation 22). For the sample problem,

$$\frac{\partial J}{\partial \mathbf{U}} = [\,-4U_2 \quad -4U_1\,] \tag{24}$$

$$\frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}} = \left[\,\frac{2U_1}{a^2} \quad \frac{2U_2}{b^2}\,\right] \tag{25}$$

Substituting into Equation 12 gives the following two equations:

$$-4U_2 + \lambda\frac{2U_1}{a^2} = 0 \tag{26}$$

$$-4U_1 + \lambda\frac{2U_2}{b^2} = 0 \tag{27}$$

which may be solved in conjunction with the equation of constraint (Equation 22) to obtain the solution, $U_1 = \frac{a}{\sqrt{2}}$, $U_2 = \frac{b}{\sqrt{2}}$ and $\lambda = 2ab$.

Solution by Method of Explicit Functions

The method of explicit functions requires a solution of Equation 14 in conjunction with the equation of constraint (Equation 22). Since there are two independent parameters, an additional equation of constraint is needed to square up the system of equations. For numerical solutions, a good choice is a function that is normal to the constraint function. A hyperbola is selected for Z_F:

$$Z_F = \frac{U_1^2}{c^2} - \frac{U_2^2}{d^2} \tag{28}$$

For the sample problem, the terms of Equation 14 are given by

$$\begin{bmatrix} \dfrac{\partial \mathbf{Z}_C}{\partial \mathbf{U}} \\ \dfrac{\partial \mathbf{Z}_F}{\partial \mathbf{U}} \end{bmatrix} = \begin{bmatrix} \dfrac{2U_1}{a^2} & \dfrac{2U_2}{b^2} \\ \dfrac{2U_1}{c^2} & -\dfrac{2U_2}{d^2} \end{bmatrix} \tag{29}$$

The required matrix inverse is

$$\begin{bmatrix} \dfrac{\partial \mathbf{Z}_C}{\partial \mathbf{U}} \\ \dfrac{\partial \mathbf{Z}_F}{\partial \mathbf{U}} \end{bmatrix}^{-1} = \frac{-1}{4U_1U_2}\left(\frac{a^2b^2c^2d^2}{a^2d^2+b^2c^2}\right)\begin{bmatrix} \dfrac{2U_2}{d^2} & \dfrac{2U_2}{b^2} \\ \dfrac{2U_1}{c^2} & \dfrac{-2U_1}{a^2} \end{bmatrix} \tag{30}$$

Substituting Equations 24 and 30 into Equation 14 yields

$$\begin{bmatrix} \dfrac{\partial J}{\partial \mathbf{Z}_C} & \dfrac{\partial J}{\partial \mathbf{Z}_F} \end{bmatrix} = \frac{-1}{4U_1U_2}\left(\frac{a^2b^2c^2d^2}{a^2d^2+b^2c^2}\right)\begin{bmatrix} \dfrac{8d^2U_1^2+8c^2U_2^2}{c^2d^2} & \dfrac{-8b^2U_1^2+8a^2U_2^2}{a^2b^2} \end{bmatrix} \tag{31}$$

The equation

$$\frac{8U_2^2}{b^2} - \frac{8U_1^2}{a^2} = 0 \tag{32}$$

is solved in conjunction with Equation 22 to obtain $U_1 = \frac{a}{\sqrt{2}}$ and $U_2 = \frac{b}{\sqrt{2}}$. The Lagrange multiplier, obtained from the first column of Equation 31, is simply $\lambda = 2ab$. Observe that at the solution point, the constants c and d completely cancel from the solution as expected verifying that Equation 28 is arbitrary.

Solution by Method of Gradient Projection

The method of gradient projection requires a solution of Equation 20 in conjunction with Equation 22. For the sample problem, U_1 is selected for $\mathbf{U}_C$ and U_2 for $\mathbf{U}_F$. Because of symmetry, the selection of which independent parameter is a “state” parameter and which is a “decision” parameter is completely arbitrary.

$$\frac{\partial J}{\partial \mathbf{U}_C} = -4U_2 \tag{33}$$

$$\frac{\partial J}{\partial \mathbf{U}_F} = -4U_1 \tag{34}$$

$$\frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_C} = \frac{2U_1}{a^2} \tag{35}$$

$$\frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_F} = \frac{2U_2}{b^2} \tag{36}$$

Substituting the above equations into Equation 20 yields

$$[-4U_1] - [-4U_2]\left[\frac{a^2}{2U_1}\right]\left[\frac{2U_2}{b^2}\right] = 0$$

$$-4U_1^2b^2 + 4U_2^2a^2 = 0 \tag{37}$$

which is solved in conjunction with Equation 22 to obtain $U_1 = \frac{a}{\sqrt{2}}$ and $U_2 = \frac{b}{\sqrt{2}}$. The Lagrange multiplier, which is also obtained as a byproduct, is given by substituting into Equation 21:

$$\lambda = -[-4U_2]\left[\frac{2U_1}{a^2}\right]^{-1} = 2ab \tag{38}$$

Second-Order Gradient Search

Parameter optimization problems with constraints where the dependent parameters are obtained by numerical integration are difficult if not impossible to solve analytically. Numerical solutions may be obtained by searching using an iterative technique like Newton's method. For the explicit functions method, the equations that need to be solved are

$$\mathbf{Z}_C = \mathbf{C} \quad \text{(M equations)} \tag{39}$$

and from the last $N - M$ columns of

$$\begin{bmatrix} \dfrac{\partial J}{\partial \mathbf{Z}_C} & \dfrac{\partial J}{\partial \mathbf{Z}_F} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial J}{\partial \mathbf{U}} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathbf{Z}_C}{\partial \mathbf{U}} \\ \dfrac{\partial \mathbf{Z}_F}{\partial \mathbf{U}} \end{bmatrix}^{-1} \tag{40}$$

the following equation is extracted:

$$\frac{\partial J}{\partial \mathbf{Z}_F} = 0 \quad \text{(N} - \text{M equations)} \tag{41}$$

From the definition of the derivative, the following difference equations may be written:

$$\Delta \mathbf{Z}_C = \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}} \, \Delta \mathbf{U} \tag{42}$$

$$\Delta \frac{\partial J}{\partial \mathbf{Z}_F} = \frac{\partial^2 J}{\partial \mathbf{U} \partial \mathbf{Z}_F} \, \Delta \mathbf{U} \tag{43}$$

The search for a solution involves finding a change in the independent control parameters that will move the current values of the constraint parameters and optimization condition to their desired values. The desired changes in the constraint parameters and optimization condition are given by

$$\Delta \mathbf{Z}_C^k = \mathbf{C} - \mathbf{Z}_C^k \tag{44}$$

$$\Delta \frac{\partial J^k}{\partial \mathbf{Z}_F} = 0 - \frac{\partial J^k}{\partial \mathbf{Z}_F} \tag{45}$$

corresponding to a change in the control parameters from $\mathbf{U}^k$ to $\mathbf{U}^{k+1}$,

$$\Delta \mathbf{U} = \mathbf{U}^{k+1} - \mathbf{U}^k \tag{46}$$

Solving for $\mathbf{U}^{k+1}$, an iterative equation is obtained for the k'th iteration:

$$\mathbf{U}^{k+1} = \mathbf{U}^k - \begin{bmatrix} \dfrac{\partial \mathbf{Z}_C^k}{\partial \mathbf{U}} \\ \dfrac{\partial^2 J^k}{\partial \mathbf{U} \partial \mathbf{Z}_F} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{Z}_C^k - \mathbf{C} \\ \dfrac{\partial J^k}{\partial \mathbf{Z}_F} \end{bmatrix} \tag{47}$$

The partial derivatives required by the optimization algorithm defined by Equation 47 are obtained by finite difference. Computation of these finite difference partial derivatives requires repeated evaluation of the functions f and g for the performance index and constraint parameters at each iteration:

$$\frac{\partial J}{\partial U_i} = \frac{f(\mathbf{U} + \Delta \mathbf{U}_i) - f(\mathbf{U} - \Delta \mathbf{U}_i)}{2 \Delta U_i} \tag{48}$$

The $\Delta\mathbf{U}_i$ vector is zero except for the i th element that contains the partial derivative step size. ΔU_i is the i'th element of $\Delta\mathbf{U}_i$. The partial derivatives of the constraint parameters with respect to the independent control parameters are given by

$$\frac{\partial Z_j}{\partial U_i} = \frac{g_j(\mathbf{U} + \Delta\mathbf{U}_i) - g_j(\mathbf{U} - \Delta\mathbf{U}_i)}{2\Delta U_i} \tag{49}$$

The elements of the required matrix of second partial derivatives are given by

$$\begin{aligned}\frac{\partial^2 J}{\partial U_j \partial Z_i} = \frac{1}{2\Delta U_j}\{ & [f(\mathbf{U} + \Delta\mathbf{U}_j + \Delta\mathbf{U}_i) - f(\mathbf{U} + \Delta\mathbf{U}_j - \Delta\mathbf{U}_i)] \\ & [g(\mathbf{U} + \Delta\mathbf{U}_j + \Delta\mathbf{U}_i) - g(\mathbf{U} + \Delta\mathbf{U}_j - \Delta\mathbf{U}_i)]^{-1} \\ & -[f(\mathbf{U} - \Delta\mathbf{U}_j + \Delta\mathbf{U}_i) - f(\mathbf{U} - \Delta\mathbf{U}_j - \Delta\mathbf{U}_i)] \\ & [g(\mathbf{U} - \Delta\mathbf{U}_j + \Delta\mathbf{U}_i) - g(\mathbf{U} - \Delta\mathbf{U}_j - \Delta\mathbf{U}_i)]^{-1}\}\end{aligned} \tag{50}$$

The partial step size for the first partial derivatives should be as small as possible to achieve linearity but large enough, relative to the machine precision, to maintain accuracy. The partial step size for the second partial derivatives ($\Delta\mathbf{U}_j$) should be about 5 to 10 times larger than the corresponding ($\Delta\mathbf{U}_i$). The computation of the second partial derivatives (Equation 50) will require $4N^2$ evaluations of the performance index and constraint functions. For six control parameters, 144 function evaluations are needed. Several methods have been explored to accelerate the computation of the second partial derivatives. Since the optimization conditions are not a function of the second partial derivatives, approximations may be used to speed up the search without compromising accuracy. An approximation that worked well for optimization of the Viking orbit insertion maneuver was to set all the terms of Equation 50 where $i \neq j$ to zero. For this approximation, $2N + 1$ function evaluations are required. Another approach, along the lines suggested by Fletcher-Powell-Davidon, was pursued in Reference 5. The matrix of second partial derivatives are initialized with an approximate solution. Subsequent changes in the control computed during the search are used to estimate and thus improve the second partial derivative matrix. This bootstrap approach can greatly speed up the search but may lead to instabilities if the search is not properly controlled.

Inequality Constraints

Sometimes the constraint on a Z parameter is not a specific target value but a range of values. In other situations, the second-order gradient search described above may not converge to the desired minimum if the initial guess required to start the search is too far from the solution but instead wander off toward a local maximum or inflection point. For these reasons, it is often convenient to specify inequality constraints where the $\mathbf{Z}$ are constrained to a specified range of values:

$$C_{Li} \leq Z_i \leq C_{Ui} \tag{51}$$

An algorithm has been devised to transform the problem of optimization with inequality constraints into the problem of optimization with equality constraints described above. At any step in the iteration for a solution, the Z_i parameters are tested and sorted into the $\mathbf{Z}_C$ category or $\mathbf{Z}_F$ category. The algorithm is diagramed on Figure 2. The following conditions result in the Z_i target variable being placed in the constrained $\mathbf{Z}_C$ category:

(a) If $C_{Li} = C_{Ui}$, C_i is set equal to C_{Li} and Z_i is a hard constraint

(b) If $Z_i > C_{Ui}$, C_i is set equal to C_{Ui} and Z_i is a soft constraint

(c) If $Z_i < C_{Li}$, C_i is set equal to C_{Li} and Z_i is a soft constraint

The following conditions result in the Z_i target variable being placed in the unconstrained $\mathbf{Z}_F$ category.

(d) If $C_{Li} < Z_i < C_{Ui}$

(e) If $|Z_i - C_{Li}| < \epsilon_b$ and $\frac{\partial J}{\partial Z_i} < 0$

(f) If $|Z_i - C_{Ui}| < \epsilon_b$ and $\frac{\partial J}{\partial Z_i} > 0$

Any of the following conditions result in convergence for a particular Z_i constraint variable. All of the constraint variables must simultaneously satisfy one of these conditions for a solution to be obtained.

(g) If $|Z_i - C_{Li}| < \epsilon_b$ and $C_{Li} = C_{Ui}$, hard constraint

(h) If $|Z_i - C_{Li}| < \epsilon_b$ and $\frac{\partial J}{\partial Z_i} > 0$, soft constraint

(i) If $|Z_i - C_{Ui}| < \epsilon_b$ and $\frac{\partial J}{\partial Z_i} < 0$, soft constraint

(j) If $C_{Li} < Z_i < C_{Ui}$ and $\frac{\partial J}{\partial Z_i} < \epsilon_p$, a true minimum satisfying the constraints

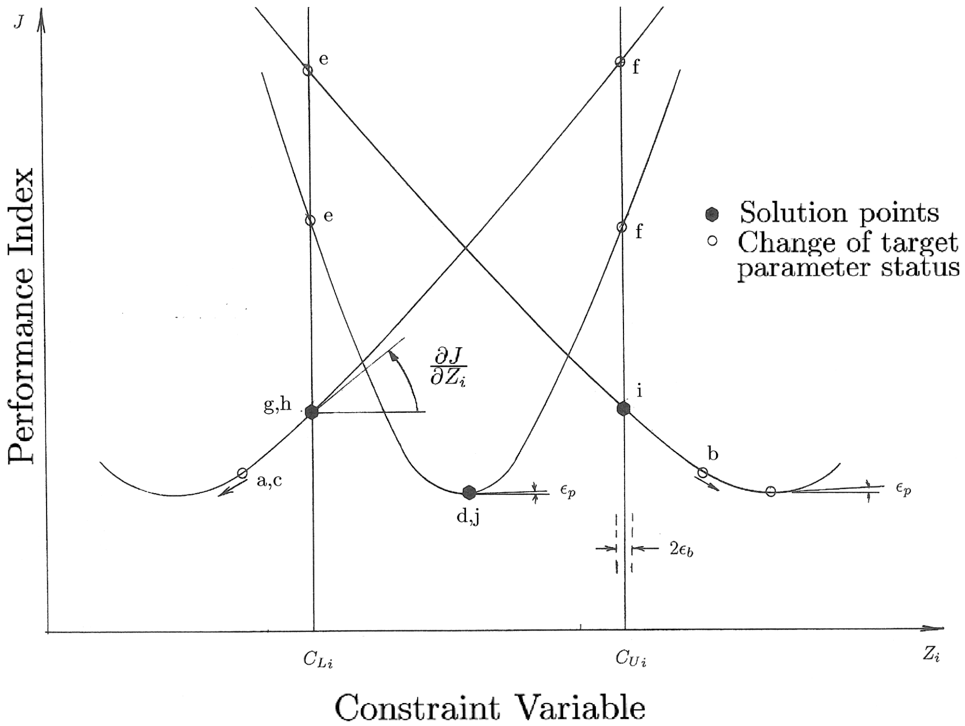

Figure 2 Inequality Constraint Status Determination

A soft constraint applies to the current iteration and may be released as the search progresses. A hard constraint is an equality constraint and applies throughout the search. The tolerance ϵ_b is on the value of the constrained variable, and the tolerance ϵ_p is on the partial derivative of J with respect to Z_i. The conditions for control of the search and confirmation of a solution are lettered a-j and shown on Figure 2. There are three possible cases that apply to each constraint variable provided the optimization problem has been properly defined and constrained. The constraint variable may either be an increasing monotone across the constraint interval, achieve a minimum within the constraint interval or be a decreasing monotone across the constraint interval. If a maximum is sought, the sign of J is changed and the algorithm searches for a minimum. These three cases are illustrated on Figure 2. For the first case, conditions (a) or (c) will select the lower bound and condition (f) will release the constraint from the upper bound. At the solution point (g,h), the partial derivative of J with respect to Z_i, the negative of the Lagrange multiplier, indicates that releasing the constraint will result in an increase in J. The solution is thus held at the lower bound. For the second case, conditions e or f will release the constraint from the lower and upper bounds, respectively, and a minimum is obtained (d,j) between the bounds. The third case is simply the mirror image of the first case.

MESSENGER Mission Example

The MESSENGER spacecraft was launched on August 3, 2004, on a mission to explore the planet Mercury. The trajectory first re-encounters Earth a year after launch, to obtain a gravity assist, and then proceeds on to several encounters with Venus and Mercury before being inserted into Mercury orbit in 2011. The initial injection error at Earth launch resulted in a 20 m/s under burn. Two Trajectory Correction

Maneuvers (TCMs) were scheduled to make up the energy deficit and place the spacecraft on the proper trajectory. Two TCMs are necessary to achieve the target; the first corrects the energy and the second corrects the orbit plane. Because of the near 360° transfer, the first maneuver, which is performed shortly after launch, is unable to correct the orbit plane. The second maneuver, which is performed about three months after launch is placed to correct orbit plane but is less efficient in correcting energy or flight time. Since there are only two constraints that need to be satisfied, the position relative to Earth in the target B-plane at the second encounter, and there are six maneuver components available to control the trajectory, the remaining four degrees of freedom may be used to minimize propellant expenditure.

The initial Earth launch injection conditions ($\mathbf{X}_0$) on August 3, 2005, were propagated to the nominal time of Earth return on August 2, 2005. Two TCMs were initially planned for August 18, 2004, and November 19, 2004. The spacecraft state at Earth return is determined by numerical integration:

$$\mathbf{X}_e = g_1(t_0,\ \mathbf{X}_0,\ t_1,\ \Delta\mathbf{V}_1,\ t_2,\ \Delta\mathbf{V}_2,\ t_e) \tag{52}$$

The maneuver velocity components, $\Delta\mathbf{V}_1$ and $\Delta\mathbf{V}_2$, are applied as finite burns at the maneuver start times t_1 and t_2. At the end time (t_e), the Cartesian state vector is transformed into hyperbolic orbit elements$\mathbf{H}_e$ with the **S** vector along the approach asymptote, **T** vector normal to **S** and in the Earth equator of J2000, and the **R** vector completing the right hand Cartesian coordinate system. The **B** vector is in the R-T plane from the center of the Earth to the intersection of the approach asymptote. The Cartesian state vector ($\mathbf{X}_e$) maps one to one into the hyperbolic elements ($\mathbf{H}_e$) given the central body gravity GM_e.

$$\mathbf{H}_e = g_2(\mathbf{X}_e, GM_e)$$

$$\mathbf{H}_e = [\mathbf{B}\cdot\mathbf{R},\ \mathbf{B}\cdot\mathbf{T}, t_p, V_\infty, \alpha_\infty, \delta_\infty] \tag{53}$$

The hyperbolic elements $\mathbf{B}\cdot\mathbf{R}$ and $\mathbf{B}\cdot\mathbf{T}$ are the coordinates of the approach asymptote in the target B-plane; t_p is the time of closest approach; V_∞ is the approach hyperbolic velocity magnitude; and α_∞ and δ_∞ are the right ascension and declination of the approach asymptote. The optimization problem is to find the velocity change components of the two TCMs that will acquire the target and minimize propellant consumption, which is related to the sum of the magnitudes of the maneuver velocity change associated with each maneuver.

The optimization problem described above must first be cast into the framework required by the optimization method being used. For all the methods described in this paper, the following constraint variables, constraint parameters, performance index and control variables are defined:

$$\mathbf{Z}_C = [\mathbf{B}\cdot\mathbf{R},\ \mathbf{B}\cdot\mathbf{T}] \tag{54}$$

$$\mathbf{C}_C = [-14,463.00\text{ km}, -17,793.00\text{ km}] \tag{55}$$

$$J = |\Delta\mathbf{V}_1| + |\Delta\mathbf{V}_2| \tag{56}$$

$$\mathbf{U} = [\Delta\mathbf{V}_{1x},\ \Delta\mathbf{V}_{1y},\ \Delta\mathbf{V}_{1z},\ \Delta\mathbf{V}_{2x},\ \Delta\mathbf{V}_{2y},\ \Delta\mathbf{V}_{2z}] \tag{57}$$

The B-plane parameters are restored to their nominal pre-launch target values, and all the other hyperbolic parameters at the second Earth flyby including flight time are permitted to float. Experience has revealed that the flight time and approach velocity errors are small enough to be corrected by subsequent maneuvers. For the method of explicit functions, four additional equations of constraint ($\mathbf{Z}_F$) must be defined. A natural choice are the equations for the four hyperbolic parameters that are not constrained:

$$\mathbf{Z}_F = [t_p,\ V_\infty,\ \alpha_\infty,\ \delta_\infty] \tag{58}$$

A problem with this choice for $\mathbf{Z}_F$ is the sensitivity of the first maneuver to parameters defined after the second maneuver. For this reason, a preliminary search is conducted with $\mathbf{Z}_F$ defined by t_p and the three velocity components of the second maneuver rotated to along track, cross track and out of plane components.

The in-plane velocity components for the second maneuver were constrained to zero, and a solution was obtained that is within 5 m/s of optimum before the search stalled because of non linearity and the approximation used for computing second partial derivatives. The search was restarted with the $\mathbf{Z}_F$ as originally described by Equation 58, and the results after each subsequent iteration are given in Table 1.

Table 1

OPTIMIZATION BY EXPLICIT FUNCTIONS

	PI	CONSTRAINTS		OPTIMIZATION CONDITION			
	J	$\mathbf{B}\cdot\mathbf{R}$	$\mathbf{B}\cdot\mathbf{T}$	$\frac{\partial J}{\partial t_p}$	$\frac{\partial J}{\partial V_\infty}$	$\frac{\partial J}{\partial \alpha_\infty}$	$\frac{\partial J}{\partial \delta_\infty}$
1	30.065554	-14462.115	-17789.846	0.1240E-05	0.3155E+00	-0.1107E+02	-0.3651E+01
2	26.249672	-14287.217	-17347.515	-0.4375E-06	-0.1903E+00	0.3855E+01	0.1314E+01
3	25.531653	-14447.957	-17750.009	-0.1150E-06	0.8816E-02	0.8356E+00	0.6061E+00
4	25.566200	-14462.791	-17792.569	-0.1677E-06	-0.2009E-01	0.1230E+01	0.8770E+00
5	25.520888	-14462.540	-17792.179	-0.4402E-07	-0.1596E-02	0.3625E+00	0.3040E+00
6	25.513694	-14462.889	-17792.492	0.2466E-07	0.3987E-02	-0.1789E+00	-0.1282E+00
7	25.530977	-14462.585	-17792.014	-0.1009E-06	-0.1068E-01	0.7454E+00	0.5284E+00
8	25.512837	-14462.740	-17792.466	-0.2351E-08	0.1684E-02	0.3485E-01	0.4391E-01
9	25.512751	-14462.996	-17792.953	0.1312E-07	0.1629E-02	-0.9681E-01	-0.6979E-01

The search algorithm attempts to drive the constraint variables to their desired values at the same time the performance index is being driven to a minimum value. At iteration 2, for example, a substantial reduction in J is achieved at the expense of driving the constraint variables away from their desired values. At iteration 4, a slight increase in performance index is obtained as the constraint variables nearly achieve their objective. From iteration 5 through 9, convergence is achieved as the optimization algorithm drives the optimization conditions to smaller values. The solution achieves an optimum within 0.1 mm/s before machine precision prohibits any further reduction. The velocity components of the two maneuvers in Earth mean equator of J2000 are

$$\Delta\mathbf{V}_1 = [12.186085, -13.684292, -8.4862428] \text{ m/s}$$
$$\Delta\mathbf{V}_2 = [3.6276212, -3.4959270, 1.7057893] \text{ m/s}$$

The first maneuver was a bit large for the maneuver system that had not been tested in space. The first maneuver was delayed until August 24, 2004, and only about 80% of the required velocity change was executed at this time. A small makeup maneuver was executed on September 24, 2004. The maneuver scheduled for November 19, 2004, was executed as planned.

The same problem may be solved by the method of gradient projection. This method requires an awkward choice of which independent parameters are "state" parameters and which are "decision" parameters. A choice of four $\mathbf{U}_F$ parameters must be made from two sets of maneuver parameters, each of dimension three. The following arbitrary partition of maneuver velocity components into the categories required by gradient projection was used for the search:

$$\mathbf{U}_C = [\Delta\mathbf{V}_{1x},\ \Delta\mathbf{V}_{1y}] \tag{59}$$

$$\mathbf{U}_F = [\Delta\mathbf{V}_{1z},\ \Delta\mathbf{V}_{2x},\ \Delta\mathbf{V}_{2y},\ \Delta\mathbf{V}_{2z}] \tag{60}$$

The gradient projection search algorithm was implemented by replacing $\mathbf{Z}_F$ by $\mathbf{U}_F$ and using the same explicit function algorithm as above. The search was started with the maneuver velocity components set to zero and the results after each iteration are given in Table 2. The first iteration moved the target variables from about 2 million km to within 20,000 km of the desired target. By the third iteration the target variables were within 200 km of their desired value and the performance index was within 1 m/s of optimum. Iterations 4-9 were within the linear region of the second partial derivatives, and quadratic convergence is observed. The indication of quadratic convergence is an order of magnitude reduction in the optimization condition after each iteration until the machine precision limit is reached.

Table 2

OPTIMIZATION BY GRADIENT PROJECTION

PI	CONSTRAINTS		OPTIMIZATION CONDITION			
J	$\mathbf{B \cdot R}$	$\mathbf{B \cdot T}$	$\frac{\partial J}{\partial \mathbf{U}_F} - \frac{\partial J}{\partial \mathbf{U}_C} \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_C}^{-1} \frac{\partial \mathbf{Z}_C}{\partial \mathbf{U}_F} = 0$			
1 0.000000	810838.303	2107832.409	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2 26.231575	-41369.351	-982.876	0.3501E-02	0.1454E+00	-0.2393E+00	-0.7324E-01
3 26.323229	-14420.254	-17938.400	-0.2132E-01	0.5700E+00	0.4645E+00	-0.1632E+00
4 25.530799	-14462.669	-17805.150	0.1309E-01	-0.3024E+00	0.9834E-01	0.1208E+00
5 25.512553	-14462.601	-17791.852	-0.3134E-02	0.4177E-01	-0.5895E-02	-0.1555E-01
6 25.512403	-14462.998	-17792.996	0.4131E-03	-0.5040E-02	0.8746E-03	0.1905E-02
7 25.512401	-14463.000	-17793.000	-0.4762E-04	0.5815E-03	-0.1010E-03	-0.2197E-03
8 25.512401	-14463.000	-17793.000	0.5495E-05	-0.6708E-04	0.1165E-04	0.2535E-04

CONCLUSION

This paper has presented a description of a constrained parameter optimization technique based on explicit functions. The optimization theory is developed and compared with other techniques currently in use. Since all of these techniques have the same mathematical foundation, they perform the same when the search for an optimum is near the solution. When far from the solution, the response to nonlinearity determines the performance. Since the method of explicit functions is more robust and provides for greater control of nonlinearity, an improvement in performance may be expected. As an example, the first two trajectory correction maneuvers of the MESSENGER mission to Mercury are optimized and the performance shown.

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