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Resonant forcing of Mercury's libration in longitude

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ABSTRACT

The period of free libration of Mercury's longitude about the position it would have had if it were rotating uniformly at 1.5 times its orbital mean motion is close to resonance with Jupiter's orbital period. The Jupiter perturbations of Mercury's orbit thereby lead to amplitudes of libration at the 11.86 year period that may exceed the amplitude of the 88 day forced libration determined by radar. Mercury's libration in longitude may be thus dominated by only two periods of 88 days and 11.86 years, where other periods from the planetary perturbations of the orbit have much smaller amplitudes.

1. Introduction

Observations of radar speckle patterns tied to the rotation of Mercury have determined that Mercury occupies Cassini state 1 with an obliquity of 2.11 ± 0.1 arcmin, and that its forced libration in longitude at a period of 88 days has an amplitude of 35.8 ± 2 arcsec (Margot et al., 2007). The large amplitude of the longitude libration coupled with the Mariner 10 determination of the gravitational harmonic coefficient C_{22} implies that Mercury has at least a partially molten core. The full dynamical, three parameter fit to the radar data shows an additional variation with a period of about 12 years that could be interpreted as either a free libration in longitude or, as we shall see, a resonant forced libration. The detection of this long period libration is only tentative, as the time span of the radar data is only about a third of the free libration period. More data are therefore required to confirm the long period variation. Whether or not the long period variation in Mercury's rotation is real, the rapid damping (time scale $\sim 2 \times 10^5$ years; Peale, 2005) and the lack of plausible excitation mechanisms make a free libration difficult to understand. An understanding of the long period libration will aid in the interpretation of the MESSENGER spacecraft determination of the orientation of the axis of minimum moment of inertia (Solomon et al., 2001). MESSENGER will determine the phase of the libration, whereas radar determines the librational angular velocity.

The perturbations of Mercury's orbital parameters by the planets include terms whose period is that of Jupiter's orbital motion. The orbital variations are transmitted to forced librations at the periods of the perturbations. Relatively large amplitude long period librations would ensue if the free libration period is close to Jupiter's orbital period. The value of $(B-A)/C_m = (2.03 \pm 0.12) \times 10^{-4}$ derived from the amplitude of the 88 day libration (Margot et al., 2007) leads to a free libration period of a little over 12 years, and the 1σ uncertainties in $(B-A)/C_m$ lead to a range of free libration periods that includes Jupiter's orbital period of 11.86 years. Here $A < B < C$ are the principal moments of inertia of Mercury, and C_m is the moment of inertia of the mantle and crust alone. The resonant forcing of Mercury's libration at Jupiter's period was noted by Margot et al. (2007) and discussed by Peale and Margot (2007). If a dominant, nearly 12 year period of variation in Mercury's libration in longitude can be identified as a resonantly forced libration from Jupiter's perturbation of the orbit, we need not seek an obscure excitation mechanism to account for an interpretation as a free libration. The libration would then not be misinterpreted in terms of Mercury's inferred dynamical history or its interior properties. Still, we have been surprised in the past, and the possibility of a free libration must be retained.

Here we include the effects of the planetary perturbations of Mercury's orbital parameters on this planet's libration in longitude, and in the process, correct an error of omission in a previous work by Peale et al. (2007). The large value of $(B-A)/C_m = 3.5 \times 10^{-4}$ used in this earlier work excluded any resonant interaction of the free libration period and Jupiter's orbital period. For the radar determined value of $(B-A)/C_m = 2.03 \times 10^{-4}$, Mercury's libration is shown to be dominated by only two periods, the 88 day forced libration that determines this parameter, and a long period libration forced at Jupiter's orbital period, where the free libration is damped to negligible amplitude. The remaining planetary perturbations of the libration can be identified with perturbations by particular planets, but they have much lower amplitudes. The am-

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plitude of the 11.86 year Jupiter induced libration would also be small were it not for the near resonant forcing. Although the amplitude of the variation in the librational angular velocity with 11.86 year period for this most probable value of $(B-A)/C_m$ is less than that of the free libration inferred in Fig. 3B of Margot et al. (2007), the latter amplitude is matched by the Jupiter forced libration for a slightly larger value of $(B-A)/C_m$ that is within the 1σ uncertainty.

The librational equations of motion are derived in Section 2 as coupled equations governing the core and mantle, and they are solved numerically in Section 3. The variations in the orbital parameters within these equations are determined from the 20,000 year JPL Ephemeris DE 408. Dissipation from tides and from the interaction of a liquid core and solid mantle is included to damp any free libration that could not be eliminated by the choice of initial conditions. The history of the libration in longitude over the 20,000 year interval covered by the ephemeris is interpreted in terms of the damping, the planetary perturbations and the secular change in Mercury's orbital parameters over the interval. The dominance of the 88 day and 11.86 year periods is demonstrated by a 30 year segment of the libration including the current date. The complete librational history is Fourier transformed to yield the power spectral density (PSD) of the various frequencies. The amplitudes of the dominant terms relative to that of the 88 day libration are determined and compared with those obtained by Dufey et al. (2008).

Details of the effect of the proximity of the free libration period to Jupiter's orbital period are given in Section 4. The equation for the free libration is derived as an average of the equations of motion over an orbit period. An approximate mantle equation is that of a damped harmonic oscillator to which we add a periodic forcing term representing the dominant term at Jupiter's orbital frequency. The amplitudes of the forced librations as determined by the approximate solution as a function of $(B-A)/C_m$ are compared with those obtained empirically as the dynamical evolution passes the current epoch. The frequency of the free librations increases with $(B-A)/C_m$ and thereby varies the nearness to resonance and the amplitude of the 11.86 year term in the libration. The relative empirical amplitudes match those of the analytic approximation very well as long as the system does not cross the resonance during the 20,000 year interval. The phases of the forced librations match those of the analytic approximation on both sides of the resonance. Two values of $(B-A)/C_m$ on opposite sides of the resonance produce the amplitude of the long period term inferred from the dynamical fit to the radar data. The libration for several values of $(B-A)/C_m$ are shown explicitly and additional consistencies of the calculated libration with the analytical approximation are pointed out. In Section 5, we compare the results directly with the radar data, where it is shown that the amplitude of the long period variation of the libration in a dynamical fit to the data can be easily accommodated within the 1σ uncertainties in $(B-A)/C_m$. We summarize our results in Section 6.

2. Rotational equations

The potential of the Sun in Mercury's gravitational field up to the second degree terms is given by (e.g., Murray and Dermott, 1999)

$$V = -\frac{GM_\odot M_M}{r}\left[1 - J_2\frac{R^2}{r^2}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + 3C_{22}\frac{R^2}{r^2}\sin^2\theta\cos 2\phi\right], \tag{1}$$

where the position of the Sun is given by the ordinary spherical polar coordinates r, θ, ϕ relative to a principal axis system

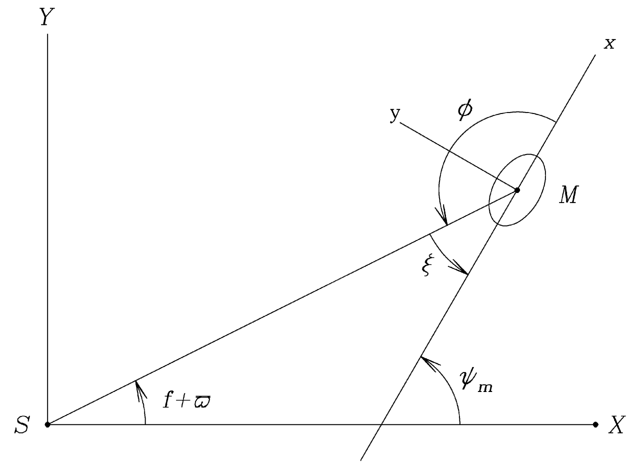

Fig. 1. Angles used for analysis of libration in longitude. SX is directed from the Sun toward the vernal equinox. M denotes Mercury with x being the axis of minimum moment of inertia. Mercury's orbit and equator planes are assumed coincident with the ecliptic.

fixed in Mercury, with the z axis coinciding with the spin axis and the x axis along the axis of minimum moment of inertia. G is the gravitational constant, $M_\odot$ and M_M are the masses of the Sun and Mercury, respectively, R is the radius of Mercury, and $J_2 = (C - A/2 - B/2)/(M_M R^2)$ and $C_{22} = (B-A)/(4M_M R^2)$ are the second degree gravitational harmonic coefficients with $A < B < C$ being the principal moments of inertia. Fig. 1 shows the geometry looking down on the plane of Mercury's orbit, where ϕ is defined explicitly and where S is the position of the Sun, the SX line is fixed along the vernal equinox of J2000, ψ_m defines the orientation of the axis of minimum moment of inertia relative to the inertial SX line, f is the true anomaly, and $\varpi = \omega + \Omega$ is the longitude of perihelion, with ω being the argument of perihelion and Ω the longitude of the ascending node of the orbit plane on the ecliptic. The angle ξ measures the orientation of the axis x of minimum moment of inertia relative to the solar direction. Since we are neglecting the variations in I, we choose the ecliptic and Mercury's equator and orbit planes to be coincident. This latter assumption will not affect the forcing of libration, since the relevant torques are perpendicular to the orbit plane whether or not that plane has its real inclination.

With $\theta \equiv \pi/2$ from our neglect of the small obliquity and the variations in the orbital inclination I, we can write the torque on the permanent distribution of mass in Mercury as the negative of the torque on the Sun in Mercury's field with $T = \partial V/\partial\phi$. Then

$$T = \frac{3}{2}\frac{GM_\odot}{r^3}(B-A)\sin 2\phi = -\frac{3}{2}\frac{GM_\odot}{r^3}(B-A)\sin 2\xi = -\frac{3}{2}\frac{GM_\odot}{r^3}(B-A)\sin 2(\psi_m - \varpi - f). \tag{2}$$

Since Mercury has a molten core (Margot et al., 2007), only Mercury's mantle and crust will respond to the external torque on the short time scales of the forced and free librations in longitude. We assume the coupling of the core to the mantle is proportional to the difference in the angular velocities of each considered as rigid bodies, which is consistent with their being coupled by a viscous fluid. In addition, solid body tides raised on Mercury lead to a torque because of the dissipation of tidal energy. Mercury's rotational equations of motion become

$$C_m\frac{d^2\psi_m}{dt^2} = -\frac{3}{2}\frac{GM_\odot}{r^3}(B-A)\sin 2(\psi_m - \varpi - f) - 3\frac{k_2}{Q_0}\frac{GM_\odot^2 R^5}{r^6}\left(\frac{\dot\psi_m}{n} - \frac{\dot f}{n}\right) - k(\dot\psi_m - \dot\psi_c),$$

$$C_c\frac{d^2\psi_c}{dt^2} = k(\dot\psi_m - \dot\psi_c), \tag{3}$$

where C_m and C_c are the moments of inertia of the mantle and core respectively, $\dot{\psi}_m$ and $\dot{\psi}_c$ the respective angular velocities, and k is a constant coupling the core to the mantle. The model for the tidal torque assumes that the equilibrium tidal bulge corresponds to the position of the sub-solar point on Mercury a short time δt in the past. This corresponds to a tidal torque where the dissipation function Q is inversely proportional to frequency, such that $\delta t = 1/Q_0 n$ where $n = \sqrt{G(M_\odot + M_M)/a^3}$ is the orbital mean motion of Mercury and Q_0 is the value of Q appropriate to the orbital frequency. [See Peale (2007) for a detailed derivation of the tidal torque.]

The core kinematic viscosity ν is related to k by equating the time constant for the decay of a differential angular velocity $\dot{\psi}_m - \dot{\psi}_c$ obtained from Eqs. (3), with all torques except that at the core–mantle boundary (CMB) set to zero, to the time scale for a fluid with kinematic viscosity ν, rotating differentially in a closed spherical container of radius R_c to become synchronously rotating with the container at angular velocity $\dot{\psi}_m$. There results $C_c C_m/[(C_c + C_m)k] = R_c/(\dot{\psi}_m \nu)^{1/2}$ (Greenspan and Howard, 1963), where $R_c \approx \beta R$ ($\beta < 1$) is the radius of Mercury's core. This time scale is appropriate for a completely molten core that participates in the induced circulation.

In determining the effect of the planetary perturbations of the orbit on Mercury's libration in longitude, Peale et al. (2007) used spline fits to JPL ephemeris DE 408 to account for the perturbations of the semimajor axis a, the eccentricity e, the longitude of the ascending node Ω and the argument of perihelion ω in the numerical solution. However, we solved for the orbital motion in cartesian coordinates, which process neglects the planetary perturbations of the true anomaly f. This omission was pointed out to us by Dufey et al. (2008). Here we correct this oversight by including the planetary perturbations of the true anomaly f as given by the ephemeris DE 408 but with the true anomaly converted to a monotonically increasing function for the spline fit. The ephemeris was sampled every 10 days in constructing the spline fits, where the latter allowed determination of the orbital elements at random times in the Burlisch–Stoer solution of the differential equations.

To make Eqs. (3) dimensionless we express the distances in AU ($a_0 = 1$ AU), scale time with the angular velocity $n_0 = \sqrt{GM_\odot/a_0^3}$ of a test particle at 1 AU ($t \rightarrow n_0 t$), which normalizes the angular velocities by n_0, and write $C = \alpha M_M R^2$. The equations become

$$\begin{aligned}\frac{d^2\psi_m}{dt^2} &= -\frac{3}{2}\frac{B-A}{r^3 C_m}\sin 2(\psi_m - \varpi - f) \\ &\quad - \frac{K_T}{r^6}\left(\frac{\dot{\psi}_m}{n} - \frac{a^2\sqrt{1-e^2}}{r^2}\right) - k'(\dot{\psi}_m - \dot{\psi}_c), \\ \frac{d^2\psi_c}{dt^2} &= k'\frac{C_m}{C_c}(\dot{\psi}_m - \dot{\psi}_c),\end{aligned} \tag{4}$$

where in dimensioned variables, $K_T = (3k_2 M_\odot R^3 C)/(\alpha Q_0 M_M a_0^3 C_m)$ and $k' = k/C_m n_0 = (1 - C_m/C)(a_0/\beta R)(\dot{\psi}_m/n_0)^{1/2}(\nu/a_0^2 n_0)^{1/2}$ with $R_c = \beta R$, and where we have used $\dot{f} = \sqrt{G(M_\odot + M_M)a(1-e^2)}/r^2$. We shall choose $\alpha = 0.34$ (Harder and Schubert, 2001) and $\beta = 0.75$ (Siegfried and Solomon, 1974) hereinafter. In Eqs. (4), distances are in AU, angular velocities are normalized by n_0, and t increases by 2π in one terrestrial year.

We substitute $r = a(1-e^2)/(1 + e\cos f)$ in Eqs. (4) to solve them numerically, where the time variation in the orbital elements a, e, f, ω, Ω from the planetary perturbations are determined from the 20,000 year JPL ephemeris DE 408 centered on calendar year 1 and sampled at 10 day intervals. The initial conditions for the angle ψ_m are chosen such that the axis of minimum moment of inertia of Mercury is oriented toward the Sun when Mercury passes its first perihelion in the ephemeris, and the initial $\dot{\psi}_m = \dot{\psi}_c$ is chosen so as to minimize as much as possible the initial amplitude of free libration discussed below. The initial amplitude of the free libration is selected by the magnitude of the deviation from $\dot{\psi}_m^0 \equiv (1.5 + \epsilon)n$, where ϵn accommodates the angular velocity due to forced librations. Equations (4) include the dissipative torques with parameters chosen to yield a damping time scale of the free libration of about 3700 years. This artificially high damping rate was chosen so that by the end of the 20,000 year time span of the ephemeris or even by calendar year 2000, the free libration amplitude that we could not completely eliminate by our choice of initial conditions is damped to negligible amplitude. The amplitudes of the planetary induced terms in the libration are essentially unaffected by the damping (Peale et al., 2007). We first determine the librational motion for the most probable value of $(B-A)/C_m = 2.03 \times 10^{-4}$ (Margot et al., 2007), and from the power spectral density of this libration, compare the amplitudes of the dominant terms to those of Dufey et al. (2008). The interesting consequences of varying $(B-A)/C_m$ within its uncertainties will then be determined.

3. Results

Fig. 2 displays the deviation of the axis of minimum moment of inertia of Mercury (x axis) from the position it would have had if the rotation were uniform at $1.5n$ over the 20,000 year interval covered by JPL Ephemeris DE 408. Damping was imposed with $k_2/Q_0 = 0.04$ and $\nu = 30$ cm^2/s leading to a damping time scale of about 3700 years. The large amplitude, long period modulation with a period near 600 years at the beginning of the plot is due to a beat frequency between a forced libration at Jupiter's orbit period and the free libration, where we were not able to quite eliminate the latter in choosing the initial conditions. (The free libration will be explained in more detail in the next section.) The free libration is damped by the imposed dissipation and its contribution to the overall libration is nearly gone by year 1. The growth in the apparent amplitude beyond year 1 is real and is due to the fact that the orbital eccentricity is growing during the 20,000 years. Small amplitude modulations due to beat periods involving the periods of the planetary perturbations of the orbit are evident in the lat-

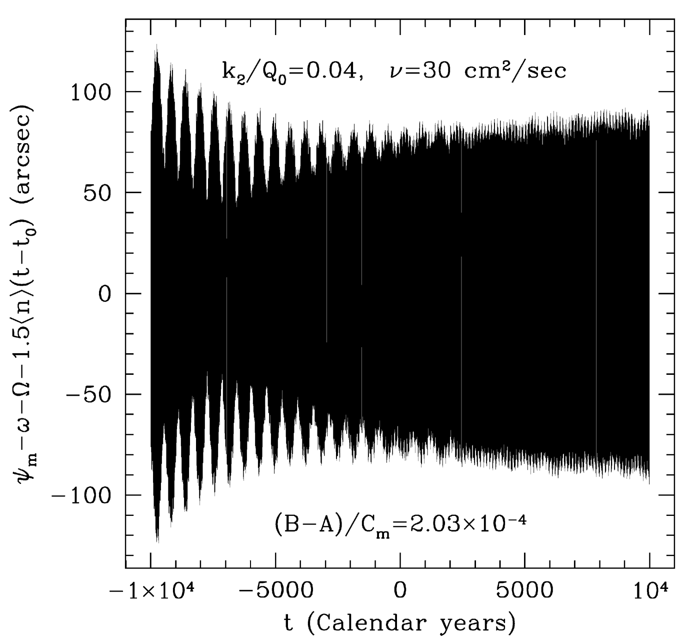

Fig. 2. Mercury's libration in longitude under the influence of the planetary perturbations of the orbit according to the 20,000 year JPL Ephemeris DE 408. Damping is applied to reduce the amplitude of the initial free libration.

Table 1
Relative power and amplitudes of the dominant peaks in the PSD shown in Fig. 4 compared with the amplitudes obtained by Dufey et al. (2008). The periods and magnitudes of the terms are determined from parabolic fits to the peaks at each frequency in the PSD. The radar determined amplitude of the 88 day forced libration is 35.8 ± 2 arcsec from which the actual amplitudes of each of the terms can be determined. The symbols in the first column correspond to the labels in Fig. 4.

	Period	Forcing argument	Power	Amplitude	Dufey et al.
1	43.98466 d	$2(\lambda - \varpi)$	0.01045	0.10223	0.11150
0	87.96935 d	$\lambda - \varpi$	1.00000	1.00000	1.00000
V	5.66316 y	$2\lambda - 5\lambda_V + 3\varpi$	9.40318×10^{-3}	0.09697	0.10691
J	5.93124 y	$2\lambda_J - 2\varpi$	1.64592×10^{-3}	0.04057	0.04111
E	6.57457 y	$\lambda - 4\lambda_E + 3\varpi$	2.2111×10^{-4}	0.01487	0.01760
J	11.86295 y	$\lambda_J - \varpi$	1.19550	1.09339	0.32611
S	14.73017 y	$2\lambda_S - 2\varpi$	2.02590×10^{-3}	0.04501	0.03030

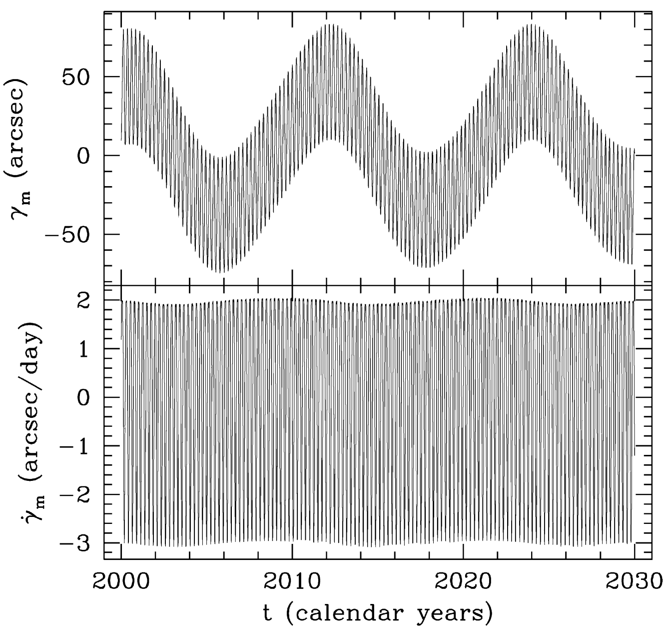

Fig. 3. Upper panel: Mercury's libration in longitude for a 30 year time interval that includes the current date. The 88 day forced libration is superposed on a long period oscillation that masquerades as a free libration. Lower panel: The deviation of Mercury's angular velocity from the resonant value of $1.5n$ showing a small amplitude long period variation corresponding to the long period oscillation in the upper panel. The ordinates $\gamma_m = \psi_m - \varpi - 1.5\langle n\rangle(t - t_0)$ and $\dot{\gamma}_m = \dot{\psi}_m - 1.5n$, where t_0 is the time of perihelion passage.

ter half of the plot. The dominant modulation beyond year 2000 reflects the 882 year beat period characteristic of the 5:2 great inequality of Saturn's and Jupiter's mutual interactions, whereas the fine scale variations result from a beat period of 124 years between the 5.66 year Venus term and the 5.93 Jupiter term in Table 1.

Fig. 3 shows a short segment of the librations in Fig. 2 along with the deviation of the angular velocity from the resonant $1.5n$. The high frequency oscillations are the 88 day forced libration due to the reversing gravitational torques from the Sun, and these are superposed on a long period variation with a period of about 12 years. This long period libration is reflected in a small long period modulation of the librational angular velocity. These latter oscillations are smaller than the tentative free libration that is consistent with a full dynamic fit to the radar data (Margot et al., 2007). However, the long period variation in Fig. 3 is not a free libration as can be seen in Fig. 4, which shows the Fourier transform (power spectral density (PSD)) of the complete libration variation shown in Fig. 2. The 88 day forced libration and its first three harmonics are numbered 0 to 3, and the other dominant frequencies are marked with V, E, J, S as being due to perturbations of the orbit by Venus, Earth, Jupiter and Saturn, respectively. The free libration frequency

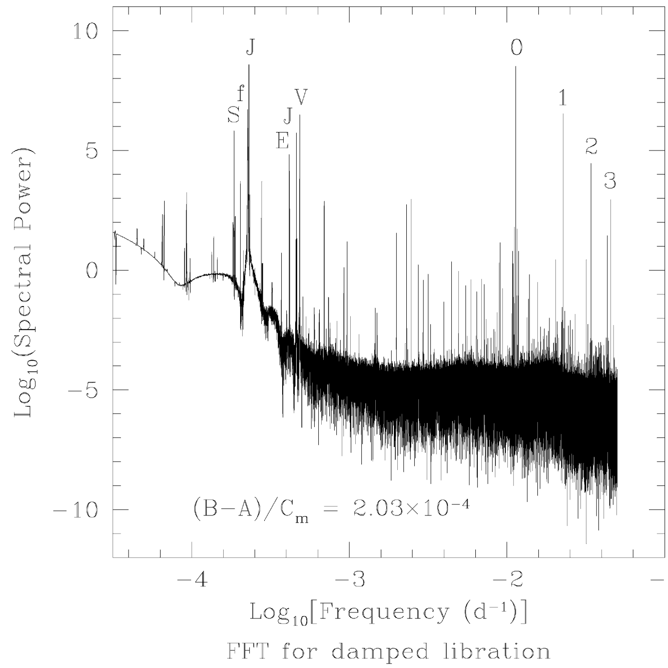

Fig. 4. Power spectral density of the libration shown in Fig. 2. The dominant forced libration period of 87.96935 days and its first three harmonics are numbered 0 to 3. The prominent frequencies are marked with V, E, J and S as being due to perturbations of the orbit by Venus, Earth, Jupiter and Saturn, respectively. The power from the free libration is denoted by f. The relative magnitudes of the planetary terms are considerably different from those obtained by Peale et al. (2007), who had neglected the planetary perturbations of f.

is denoted by f. The Jupiter contribution to the PSD at a period of 11.86 years is about the same as that of the 88 day forced libration, which is consistent with the approximately 12 year oscillation in the top panel of Fig. 3 being about the same amplitude as that of the 88 day libration superposed on it.

To complete the discussion of the PSD in Fig. 4, we have constructed in Table 1 the ratios of the magnitudes of the power densities of the dominant frequencies to that of the 88 day libration designated by 0. The line centers and actual peaks are determined by passing a parabola through the maximum and the two adjacent points on either side for each frequency. The ratio of the amplitudes of the terms in the libration spectrum is then the square root of the power density ratios. This gives a reasonably accurate ratio of the amplitudes as the FWHM of the different lines are comparable. Finally, the amplitudes are compared with those of Dufey et al. (2008), where there is reasonably good agreement except for the 11.86 year Jupiter term. The reason for this difference is that Dufey et al. use a value of $C_{22} = 1.0 \times 10^{-5}$ and our values of $C/M_M R^2 = 0.34$ and $C_m/C = 0.5$ leading to $(B - A)/C_m = 2.35 \times 10^{-4}$ instead of 2.03×10^{-4}. Why this makes a difference in the ratio of the amplitudes of 11.86 year and 88 day terms in Mercury's libration, and why the Jupiter term is so relatively large in the first place are explained in the next section.

4. Details of the free libration effects

Since Mercury's angular velocity is on average $1.5n$, we can study the long period librations about the resonant value by writing $\dot{\psi}_m = 1.5n + \dot{\gamma}_m$, such that $\psi_m = 1.5M + \varpi + \gamma_m$, where $M = n(t - t_0)$ is the mean anomaly with t_0 being the time of perihelion passage. The constant of integration is chosen such that γ_m measures the angle between the axis of minimum moment and the direction to the Sun when Mercury is at perihelion. The argument of the sine in Eq. (3) is then $2(\gamma_m + 1.5M - f)$. The angle γ_m is a slowly varying quantity, and so its motion can be studied by expanding terms like $(a^k/r^k)(\cos f, \sin f)$ in terms of the mean anomaly M (e.g., Murray and Dermott, 1999) and averaging Eq. (3) over an orbit period while holding γ_m constant. The tidal torque is averaged directly without an expansion. There results for the relative motion of the mantle,

$$\ddot{\gamma}_m + 3n^2\frac{B-A}{C_m}G_{201}(e)\gamma_m + \left(\frac{F}{C_m n} + \frac{k}{C_m}\right)\dot{\gamma}_m = \frac{k}{C_m}\dot{\gamma}_c - \frac{FD}{C_m} + \left[\frac{E}{C_m}\cos wt\right], \quad (5)$$

where the second term results from the resonant terms in expansions of $(a^3/r^3)(\cos f, \sin f)$ in the mean anomaly, all other terms in the expansion averaging to zero, and where we have used $\sin 2\gamma_m \approx 2\gamma_m$. $G_{201}(e) = 7e/2 - 123e^3/16 + 489e^5/128 - \cdots$ is an infinite series with the Kaula (1966) notation. The averaged tidal torque is of the form $\langle T_T\rangle = -F(D + \dot{\gamma}/n)$ where $F = 3k_2 n^4 R^5 f_2(e)/Q_0 G$ and $D = 1.5 - f_1(e)/f_2(e)$ with $f_1(e) = (1 + 15e^2/2 + 45e^4/8 + 5e^6/16)/(1-e^2)^6$, and $f_2(e) = (1 + 3e^2 + 3e^4/8)/(1-e^2)^{9/2}$ (e.g., Peale, 2007). The variables γ_m, γ_c, $\dot{\gamma}_m$, $\dot{\gamma}_c$ are now averages around the orbit. The last term in square brackets on the right hand side is an added external forcing term at the frequency of Jupiter's orbit, which is appropriate because that term in PSD is the dominant long period forcing of the libration in longitude.

In general, $|\dot{\gamma}_c| \ll |\dot{\gamma}_m|$, since it is only weakly coupled to the mantle, and its neglect means Eq. (5) is simply the equation of a damped harmonic oscillator forced at frequency w with natural radian frequency $w_0 = n\sqrt{3(B-A)G_{201}(e)/C_m}$ and damping constant $b = F/C_m n + k/C_m$, whose well known solution is

$$\gamma_m = \exp\left(\frac{-bt}{2}\right)D_1'\cos(w_0't + \phi_1) + \frac{E\cos(wt + \phi_2)}{C_m\sqrt{(w_0^2 - w^2)^2 + w^2b^2}}, \quad (6)$$

where $w_0' = \sqrt{w_0^2 - b^2/4}$, D_1' and ϕ_1 are determined by initial conditions, and $\sin\phi_2 = -wb/\sqrt{(w_0^2 - w^2)^2 + w^2b^2}$; $\cos\phi_2 = (w_0^2 - w^2)/\sqrt{(w_0^2 - w^2)^2 + w^2b^2}$. Without the forcing term or the damping, the angle γ_m would librate about zero with frequency w_0, which means that if one could view Mercury only at the times it passed perihelion, the axis of minimum moment of inertia would slowly swing back and forth about the direction to the Sun at frequency w_0. For the best fit radar value of $(B-A)/C_m = 2.03 \times 10^{-4}$, $2\pi/w_0 = 12.0655$ years (Fig. 5). Since the amplitude and phase of this libration are arbitrary, we call this a free libration. For plausible values of $k_2/Q_0 = 0.004$ and $\nu = 0.01$ cm^2/s, the free libration is damped with a time scale near 2×10^5 years (Peale, 2005).

The relevance of Eq. (6) to the current libration state of Mercury comes from the fact that Jupiter's orbital period is close to the period of free libration $2\pi/w_0$, and the amplitude of the libration at this frequency can therefore be abnormally large. Fig. 5 shows the free libration period as a function of $(B-A)/C_m$ for the current value of the orbital eccentricity $e = 0.20563$. The free libration period corresponding to the radar determined value of $(2.03 \pm 0.12) \times 10^{-4}$ is indicated at a little longer than 12 years on the curve along with the $\pm 1\sigma$ values of the period. Jupiter's orbital period (11.86 years) is also indicated on the curve, and it falls within the one sigma uncertainty in the free libration period. Now we can understand why the frequency of the Jupiter orbital motion is so dominant in the PSD shown in Fig. 4. The most probable value of $(B-A)/C_m = 2.03 \times 10^{-4}$ leads to w_0 being relatively close to w, such that the coefficient of $\cos(wt + \phi_2)$ in Eq. (6) is relatively large. The increase in e from 0.20318 to 0.20699 during the 20,000 year interval of the DE 408 ephemeris decreases $2\pi/w_0$ and brings the period closer to the Jupiter period in Fig. 5.

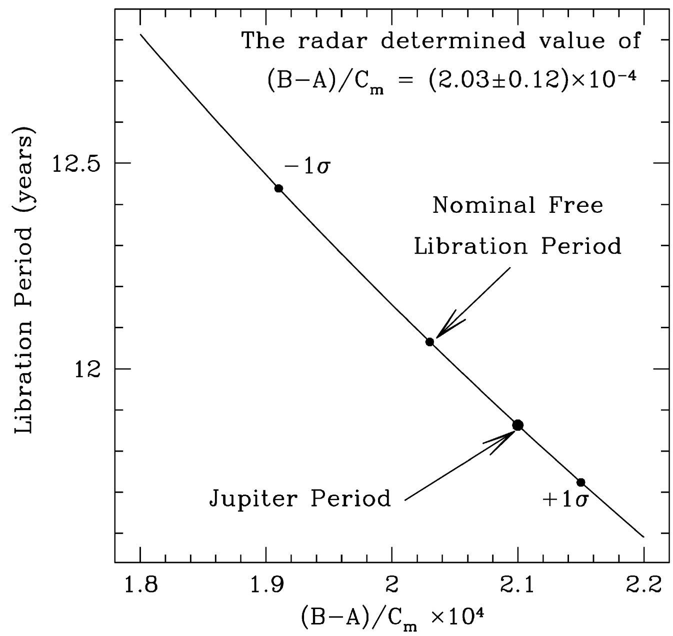

Fig. 5. Free libration period for Mercury as a function of $B - A/C_m$ with the nominal period and its 1σ extremes relative to Jupiter's orbit period indicated by the dots.

The resulting growth in the coefficient of $\cos(wt + \phi_2)$ in Eq. (6) explains why the amplitude of the libration grows after the free libration is damped to negligible amplitude in Fig. 2.

Radar determines the librational angular velocity of Mercury, so we plot the amplitude of the derivative of the last term in Eq. (6) as a function of $(B-A)/C_m$ in Fig. 6, where the coefficient E is determined such that amplitude of the forced libration at the 11.86 year period matches that obtained numerically when $w_0 \ll w$. Empirical measures of the amplitude of the 11.86 year term in the variation of the differential angular velocity were determined by integrating Eq. (3) over the 20,000 year interval for various values of $(B-A)/C_m$ and determining the long period amplitude from year 2000 to 2030 as in the lower panel of Fig. 3. These empirical amplitudes appear as dots in Fig. 6 and are seen to match the analytic approximation of the amplitude very well for the smaller values of $(B-A)/C_m$ but a few points at or just on the other side of the resonance differ substantially. This difference is due to the fact that the system is carried across the resonance by the growing eccentricity during the integration and the 30 year interval beginning at year 2000 fell at varying stages of this traverse for these values of $(B-A)/C_m$, where the system had not completely relaxed to a steady state. The dissipation here reduces the amplitudes from values appropriate to more realistic values (much smaller) of the dissipation parameters. For example, for $(B-A)/C_m = 2.095 \times 10^{-4}$ that reduction is about 11%, which is determined by evaluating the coefficient of $\cos(wt + \phi_2)$ in Eq. (6) with and without $b = 0$. The reduction in amplitude is less than 11% for smaller values of $(B-A)/C_m$. The reduction in amplitude is also less than 11% for $(B-A)/C_m > 2.105 \times 10^{-4}$ on the other side of the resonance that occurs at $(B-A)/C_m = 2.100 \times 10^{-4}$ (Fig. 5).

The change in w_0 with $(B-A)/C_m$ also explains why the Dufey et al. value for the amplitude of the Jupiter 11.86 year term in Table 1 is a factor of 0.298 less than the value we obtain. Their choice of $(B-A)/C_m = 2.35 \times 10^{-4}$ leads to $w_0 = 1.775 \times 10^{-8}$ s^{-1} that is on the other side of the frequency $w = 1.678 \times 10^{-8}$ s^{-1} from our value $w_0 = 1.650 \times 10^{-8}$ s^{-1} ($e = 0.20563$). For the parameters $k_2/Q_0 = 0.04$ and $\nu = 30$ cm^2 s^{-1}, $b = 1.703 \times 10^{-11}$. Substitution of these frequencies into the expression for the amplitude of the $\cos(wt + \phi_2)$ term in Eq. (5) gives a ratio of forced amplitudes of

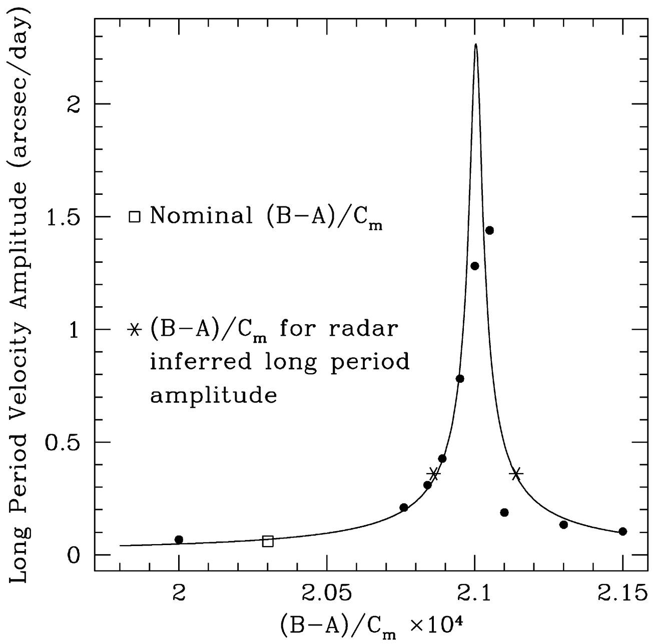

Fig. 6. Analytic estimate (solid line) of the amplitude of Mercury's librational angular velocity forced at Jupiter's orbital period as a function of the proximity of the free libration frequency w_0 to Jupiter's orbital frequency w as determined by the value of $(B-A)/C_m$. Dots indicate values of the amplitudes obtained numerically with the meaning of the special symbols indicated in the figure.

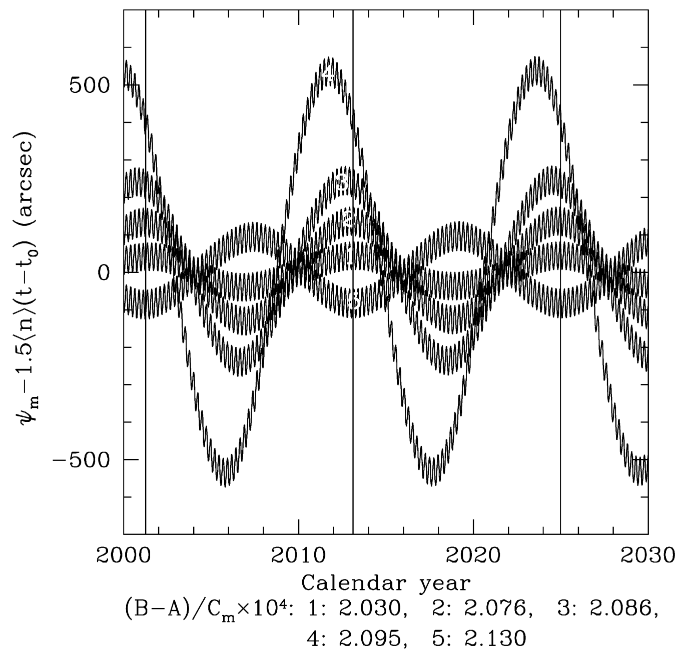

Fig. 7. Phases and amplitudes of Mercury's libration in longitude with dominant long period oscillation with an 11.86 year period. The vertical lines indicate Jupiter's passage of Mercury's perihelion longitude ($\lambda_J = \varpi$). The amplitudes and phases are consistent with the analytic approximation to the motion. (See Fig. 8.)

0.278, which is in reasonable agreement with the ratio from Table 1. For both values of w_0, $wb \ll |w_0^2 - w^2|$ so the imposed dissipation has little effect on either of these amplitudes of libration in longitude forced by the 11.86 year Jupiter term.

In the top panel of Fig. 3, the amplitude of the 88 day forced libration is $\sim 36''$, consistent with the radar value of $(35.8 \pm 2)''$, and the deduced $(B-A)/C_m = (2.03 \pm 0.12) \times 10^{-4}$ (Margot et al., 2007) that we used for that figure. The amplitude of the long period modulation in this same panel is $\sim 40''$, which is close to the $1.09339 \times 35.8'' = 39.14''$ inferred from Table 1.

In addition to the amplitudes of the forced oscillation matching those of the analytic approximation, at least for $w_0 < w$, the phases of the long period oscillation are also consistent. Given that the argument of the term in Mercury's disturbing function due to Jupiter is $\lambda_J - \varpi$, one expects the phase of the 11.86 year libration in longitude to be related to Jupiter's passage by the Mercury perihelion or aphelion longitude. Mercury's libration in longitude for 30 years spanning the present for several values of $(B-A)/C_m$ is shown in Fig. 7. The times when Jupiter passes Mercury's perihelion position of $\varpi = 77.465°$ relative to the vernal equinox of J2000 are indicated by the vertical lines. The amplitudes display the expected behavior of increasing with increasing values of $(B-A)/C_m$ yielding values of w_0 closer to w, and the phases show the consistent qualitative behavior of the minimum of the long period oscillations having a phase lag closer to $-180°$ from the times of Jupiter's passage of Mercury's perihelion longitude ($\lambda_J - \varpi = 0$) for values of w_0 further below w. In addition, there is a reversal in phase for $(B-A)/C_m = 2.130 \times 10^{-4}$ as expected, because the corresponding value of w_0 is now larger than w. Fig. 8 shows the phase shift of the libration from the forcing phase wt determined analytically (solid line) as a function of $(B-A)/C_m$ for the assumed values of $k_2/Q_0 = 0.04$ and $\nu = 30\ \mathrm{cm}^2/\mathrm{s}$. [For more realistic values of these parameters, the transition from values near $-180°$ to values near $0°$ would be more abrupt (closer to a step function transition).] From Fig. 7 we can determine an approximate negative phase shift of the long period minimum of each curve relative to the zero of the forcing argument (when Jupiter passes Mercury's perihelion longitude). With phase advancing 360° in the 11.86 year Jupiter period, these phase shifts in degrees are represented by dots in Fig. 8 for the values of $(B-A)/C_m$ for which the oscillations are plotted in Fig. 7 plus two additional extreme values of $(B-A)/C_m$. The empirical phase shifts match those of the analytic estimate within the uncertainty of their determination, which uncertainty includes shift in the precise position of the minima in Fig. 7 from the other planetary perturbations. For more realistic choices of the damping parameters, the phase of the resonant forced libration will be such that either the long period librational motion will be maximal when Jupiter passes Mercury's

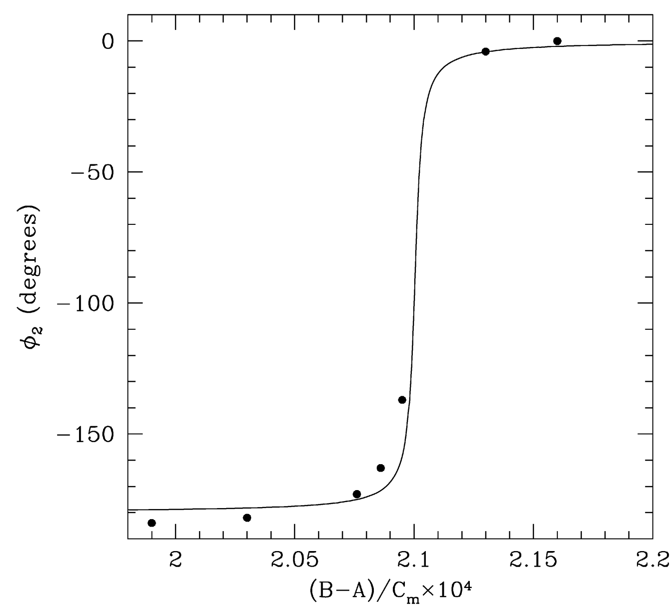

Fig. 8. The solid line is the phase shift ϕ_2 of Mercury's longitude libration forced at Jupiter's orbital period according to the approximate solution of Eq. (6). The dots are the displacements in phase of the first minimum of the long period variations in Fig. 7 from the time when $\lambda_J - \varpi = 0$.

perihelion ($w_0 < w$) or minimal ($w_0 > w$), with the amplitude being determined by the value of $(B-A)/C_m$. This contrasts with a free libration which would have an arbitrary amplitude and phase.

5. Comparison with the radar data

In Fig. 3B of Margot et al. (2007), the slope of the mean value of the differential angular velocity is consistent with a free libration in longitude as we have defined it above. Since the free libration damps on a time scale of about 10^5 years (Peale, 2005), a finite amplitude free libration would be hard to understand, as there are no obvious candidates for a recent excitation. But it is not consistent with a forced libration at Jupiter's orbital frequency whose amplitude is enhanced by the proximity of the free libration period to Jupiter's orbital period. If we infer that the long period variation in Fig. 3B of Margot et al. (2007) is a forced libration at Jupiter's orbital period and that the extreme right of the curve is a minimum, the drop in the mean value of the librational angular velocity of $0.6''$/d in 18 Mercury orbital periods implies a forced variation in the mean librational angular velocity with amplitude $0.360''$/d and a libration amplitude of $249''$. These values are considerably larger than the $0.06''$/d and $40''$ seen in the bottom and top panels of Fig. 3, but they would be produced if $(B-A)/C_m = 2.086 \times 10^{-4}$ or 2.114×10^{-4}, where both values are within the 1σ uncertainty. The amplitudes of the librational angular velocity for these values of $(B-A)/C_m$ along with the nominal value are indicated in Fig. 6 with distinct symbols.

If we consider the librational angular velocities for the two values of $(B-A)/C_m$ that yield long period amplitudes comparable to that for the best 3 parameter fit to the radar data, we find in Fig. 9 that neither matches the phase of the radar librational angular velocity. This mismatch is not surprising considering the short time span of the radar data and the tentative nature of the long period librations implied by the best 3 parameter fit. We expect to find a significantly different long period component with more data. The high sensitivity of the long period libration amplitude to the value of $(B-A)/C_m$ and the constraint on the phase provides a useful consistency check to the best fit value determined by the amplitude of the 88 day libration, and may ultimately be used to reduce the uncertainty in $(B-A)/C_m$.

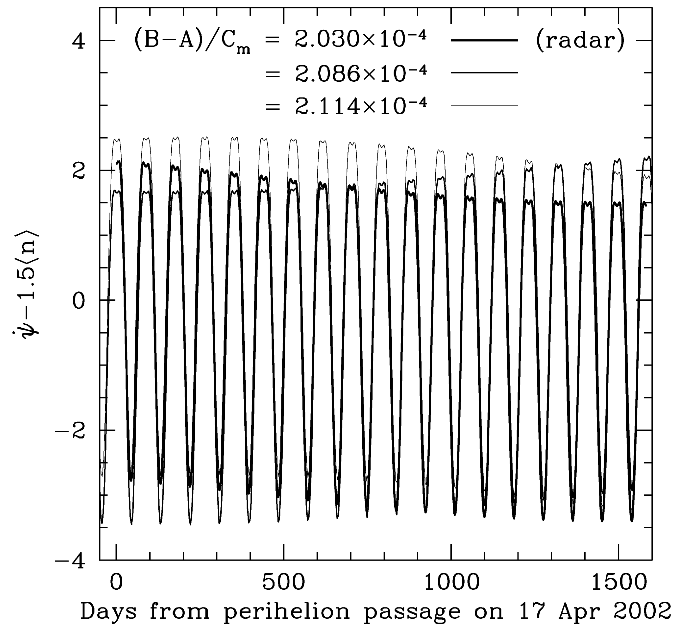

Fig. 9. Comparison of the three parameter, dynamical fit of the radar measurements of the librational angular velocity treated as a free libration, with the forced libration at Jupiter's orbital frequency, for values of $(B-A)/C_m$ that yield the same amplitude of the long period variation while remaining within the 1σ uncertainty of the moment difference ratio. The two forced librations are of nearly opposite phase because the two values of $(B-A)/C_m$ yield values of free libration frequency w_0 on opposite sides of the resonance with the Jupiter orbital frequency w.

6. Summary

The main result of this paper is the fact that Mercury should have a forced libration at Jupiter's orbital period whose amplitude may exceed that of the 88 day forced libration. This assertion depends on the condition that the value of $(B-A)/C_m$ remain close to its currently most probable value as additional observational data is accumulated. The forced long period libration will be distinguished from a free libration of comparable period by having a definite phase and amplitude, whereas a free libration phase and amplitude will be arbitrary. Tidal and core–mantle dissipation should have reduced the free libration to negligible amplitude, but we might be surprised.

We have solved the equations of rotational motion to determine Mercury's libration in longitude, while including the variations in the orbital parameters, a, e, f, ω, Ω as given by the 20,000 year JPL Ephemeris DE 408. Variations in the orbital inclination are neglected, since these cannot affect the libration in longitude in a significant way. Dissipation in the form of tides and a viscous coupling between a solid mantle and liquid core are included to damp the free libration to negligible amplitude after about 10,000 to 12,000 years except when the value of the free libration frequency w_0 crosses the resonance with Jupiter's orbital frequency w during the 20,000 years. The square root of the ratio of several dominant peaks in the PSD due to Venus, Earth, Jupiter and Saturn to the dominant peak at a period of 88 days yields the ratio of the amplitudes of these planetary caused terms to the amplitude of the 88 day libration. The measured value of the latter amplitude of $(35.8 \pm 2)''$ (Margot et al., 2007) allows determination of the actual amplitudes of the selected librational terms. These amplitude ratios are considerably different from those obtained by Peale et al. (2007), because they had omitted the planetary perturbations of the true anomaly f. The ratios are now consistent with those of Dufey et al. (2008), which were obtained through a clever Hamiltonian analysis.

The behavior of Mercury's libration is analogous with that of a damped harmonic oscillator forced at a frequency w (Jupiter's orbital frequency) near its resonant frequency w_0 (free libration frequency). Amplitudes of the 11.86 year libration increase as w_0 approaches w, and the phase of the libration reverses for w_0 on opposite sides of the resonance. The phase lags of the long period variation in libration follow the trend in the analytic approximation. The phase of the forced libration for either value of $(B-A)/C_m$ within the 1σ uncertainty that produces an amplitude of 11.86 year libration like that inferred from a dynamic fit to the radar data does not agree with the phase of the libration in that fit. This emphasizes the tentative nature of the long period libration signature as determined by radar and the need for more observational data to secure the details of the long period libration. The sensitivity of the long period libration amplitude to the proximity of w_0 to w, might be used to reduce the uncertainty in the radar determination of $(B-A)/C_m$ obtained from the amplitude of the 88 day libration.

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